Massive Relativistic Particle Model with Spin and Electric Charge from Two-Twistor Dynamics

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Abstract

The sixteen real coordinates of two-twistor space are transformed by a nonlinear mapping into an enlarged space-time framework. The standard relativistic phase space of coordinates (X_μ, P_μ) is supplemented by a six-parameter spin phase manifold (two pairs $(\eta_\alpha, \sigma_\alpha)$ and $(\overline{\eta}_{\dot\alpha}, \overline{\sigma}_{\dot\alpha})$ of canonically conjugated Weyl spinors constrained by two second class constraints) and the electric charge phase space (e, ϕ). The free two-twistor classical mechanics is rewritten in this enlarged relativistic phase space as a model for a relativistic particle. Definite values for the mass, spin and the electric charge of the particle are introduced by means of three first class constraints.

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1 Introduction

Twistor theory (see *e.g.* [1]–[5]) offers an alternative geometric picture of space-time physics based on the following basic propositions:

i) The basic geometry is spinorial and conformal *i.e.*, the fundamental elementary objects are massless.

ii) Space-time points as well as the momentum and other generators of the conformal symmetries are composite, and given in terms of fundamental conformal spinors -the twistors.

iii) The appearance of mass, spin and charge of the elementary object is a result of a composite twistor structure; the mass parameter breaks the conformal invariance down to the Poincaré one.

One of the important tasks is to translate the multitwistor (in particular two-twistor) geometry into an extended space-time framework, a step that only recently has been completed [6]–[8]. Indeed, in the standard Penrose approach (see *e.g.* [1]–[5]) space-time points are given by the relation

$$z^{\alpha\dot\beta} = \frac{i}{f}\left(\omega^\alpha\overline{\eta}^{\dot\beta} - \lambda^\alpha\overline{\pi}^{\dot\beta}\right), \qquad\qquad z^{\alpha\dot\beta} = \frac{1}{2}\sigma_\mu^{\alpha\dot\beta}z^\mu, \qquad (1.1)$$

where

$$f = \overline{\pi}^{\dot\alpha}\overline{\eta}_{\dot\alpha}, \qquad (1.2)$$

Expression (1.1) describes the composite complex Minkowski coordinates $z_\mu = x_\mu + iy_\mu$ as a solution of the two Penrose incidence equations [1]–[5]

$$\omega^\alpha = iz^{\alpha\dot\beta}\overline{\pi}_{\dot\beta}, \qquad \lambda^\alpha = iz^{\alpha\dot\beta}\overline{\eta}_{\dot\beta}, \qquad (1.3)$$

involving two twistors Z_i^A ($A = 1,\ldots,4;\ i = 1,2$) defined by two pairs of complex Weyl spinors,

$$Z_1^A = (\omega^\alpha, \overline{\pi}_{\dot\alpha}), \qquad Z_2^A = (\lambda^\alpha, \overline{\eta}_{\dot\alpha}). \qquad (1.4)$$

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The non-zero fundamental twistorial Poisson brackets (TPB) are given by the holomorphic relations

$$\left\{\pi_\alpha, \omega^\beta\right\} = i\delta_\alpha^\beta \quad , \qquad \left\{\eta_\alpha, \lambda^\beta\right\} = i\delta_\alpha^\beta , \tag{1.5a}$$

$$\left\{\overline{\pi}_{\dot\alpha}, \overline{\omega}^{\dot\beta}\right\} = -i\delta_{\dot\alpha}^{\dot\beta} \quad , \qquad \left\{\overline{\eta}_{\dot\alpha}, \overline{\lambda}^{\dot\beta}\right\} = -i\delta_{\dot\alpha}^{\dot\beta} , \tag{1.5b}$$

They correspond to the two-twistor symplectic two-form ($i = 1, 2$)

$$\Omega = d\Theta = idZ_i^A \wedge d\overline{Z}_{Ai} , \tag{1.6}$$

where Θ is the Liouville one-form, that may be expressed by

$$\Theta = \frac{i}{2} \left(Z_i^A d\overline{Z}_{Ai} - \overline{Z}_i^A dZ_{Ai} \right) . \tag{1.7}$$

Now comes a crucial point: using the relations (1.1), (1.5a) and (1.5b) one can calculate the TPB of the real composite Minkowski coordinates $x_\mu = Rez_\mu$. It turns out (see [9]) that

$$\{x_\mu, x_\nu\} = -\frac{1}{m^4} \, \epsilon_{\mu\nu\rho\tau} W^\rho P^\tau , \tag{1.8}$$

where

$$P^\mu = \sigma^\mu_{\alpha\dot\beta} \left(\pi^\alpha \overline{\pi}^{\dot\beta} + \eta^\alpha \overline{\eta}^{\dot\beta} \right) , \tag{1.9a}$$

$$W^\mu = \sigma^\mu_{\alpha\dot\beta} \left[k \left(\pi^\alpha \overline{\pi}^{\dot\beta} - \eta^\alpha \overline{\eta}^{\dot\beta} \right) + \rho\eta^\alpha \overline{\pi}^{\dot\beta} + \overline{\rho}\pi^\alpha \overline{\eta}^{\dot\beta} \right] , \tag{1.9b}$$

and

$$\rho = \omega^\alpha \eta_\alpha + \overline{\lambda}^{\dot\alpha} \overline{\pi}_{\dot\alpha} , \tag{1.9c}$$

$$\overline{\rho} = \overline{\omega}^{\dot\alpha} \overline{\eta}_{\dot\alpha} + \lambda^\alpha \pi_\alpha , \tag{1.9d}$$

$$k = \overline{k} = \frac{1}{2} \left(\omega^\alpha \pi_\alpha + \overline{\omega}^{\dot\alpha} \overline{\pi}_{\dot\alpha} - \lambda^\alpha \eta_\alpha - \overline{\lambda}^{\dot\alpha} \overline{\eta}_{\dot\alpha} \right) . \tag{1.9e}$$

Thus, the TPB of the space-time coordinates is non-zero, *i.e.* after quantization the composite space-time coordinates will become *non-commutative.*

The composite fourvector (1.9b) may be identified with the Pauli-Lubański vector that describes spin in an arbitrary relativistic frame. It is orthogonal, as it should, to the momentum (1.9a),

$$P^\mu W_\mu = 0 . \tag{1.10}$$

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The composite structure of W^μ in the primary twistor variables is further exhibited by expressing k, ρ and $\bar{\rho}$ in Eq. (1.9b) as

$$k = \frac{1}{2}\left(t_{11} - t_{22}\right), \qquad \rho = t_{12}, \qquad \bar{\rho} = t_{21}, \qquad (1.11\text{a})$$

where

$$t_{ij} = \tau^a_{ij}\, t^a = Z^A_i \overline{Z}_{Aj} \quad (a = 0,1,2,3;\ i,j = 1,2;\ A = 1,\ldots,4). \quad (1.11\text{b})$$

The four isospin Pauli matrices τ^a describe four conformal, $U(2,2)$-invariant scalar products in two-twistor space $T \otimes T$. The t^a satisfy the $u(2) = su(2) \oplus u(1)$ twistorial Poisson algebra brackets $(r,s,u = 1,2,3)$

$$\{t^r, t^s\} = \epsilon^{rsu}\, t^u, \qquad \{t^0, t^r\} = 0, \qquad (1.12)$$

as it follows from Eqs. (1.5a) and (1.5b).

The non-commutativity of the space-time coordinates in the presence of nonvanishing spin $(W_\mu \neq 0)$ can be traced back to earlier considerations (see e.g. [10, 11]). We recall, however, that the usual classical and quantum relativistic free fields are defined on a classical, commutative space-time. In this paper, following [6]–[8], we shall show how to replace the non-commutative composite coordinates x_μ by commutative ones X_μ (see Sect. 2, (2.9a) and (2.9b)). In this way, we obtain a standard relativistic phase space with TPB

$$\{X_\mu, X_\nu\} = 0, \qquad \{P_\mu, P_\nu\} = 0, \qquad \{P_\mu, X_\nu\} = \eta_{\mu\nu}, \qquad (1.13)$$

with X_μ and P_μ constructed out of twistor primary coordinates.

Our aim in this letter is to provide the geometric basis for the formulation of two-twistor dynamics in terms of more physical variables, describing the extended commutative space-time framework. In Sect. 2 we show that the 16-dimensional two-twistor phase space, described by the symplectic two-form (1.6) (or the TPB's (1.5a) and (1.5b)), is mapped bijectively on the relativistic phase space (X_μ, P_μ) enlarged by an eight-dimensional manifold M_8 providing the values of the mass, the spin and the electric charge. It turns out that this additional manifold M_8 may also be regarded as a subset of a ten-dimensional symplectic vector space spanned by two Weyl spinors and two scalars $Y_k = [(\eta_\alpha, \overline{\eta}_{\dot{\alpha}}, \sigma_\alpha, \overline{\sigma}_{\dot{\alpha}}, e, \phi);\ \alpha, \dot{\alpha} = 1,2;\ k = 1\ldots10]$ satisfying two Poincaré-invariant second class constraints R_1, R_2, the first one depending on the four-momentum. We define the symplectic structure on M_8 by means of the constrained variables Y_k satisfying the appropriate Dirac brackets.

In Sect. 3 we introduce an action for a relativistic particle model which is inspired by the two-twistor conformal-invariant Liouville one-form (1.7). The eighteen-dimensional space (X_μ, P_μ, Y_k) is restricted by the two second class constraints $R_1 = R_2 = 0$ as well as by three additional first class constraints, corresponding after first quantization to the wave equations describing mass, spin and electric charge. It appears that in such a Lagrangian formulation we reproduce the twistor Poisson brackets for composite coordinates (X_μ, P_μ) as the Dirac brackets.[1]

We would like to point out that in our formulation all eight relativistic phase space coordinates (X_μ, P_μ) satisfying the TPB relations (1.13) are two-twistor composites. This last remark is relevant in order to distinguish the present approach from earlier attempts to describe the spin degrees of freedom of free massive relativistic particles in terms of spinorial and twistorial coordinates (see *e.g.* [12]–[14]), in which the space-time manifold was introduced as a primary geometric object.

The framework presented in this paper may be considered as a generic one. Indeed, it can be further extended by adding supersymmetries (see *e.g.* [15, 16]) as well as by going to higher ($D > 4$) dimensions. In particular, we would like to notice here that the $D = 11$ BPS preons, introduced in [17] as fundamental constituents of BPS states in M-theory, may be described in terms of $D = 11$ generalized supertwistors. Finally, two-twistor space provides a natural framework for the introduction of infinite-dimensional higher spin multiplets (see [18]), with arbitrary masses and charges.

2 From two-twistor phase space to enlarged relativistic phase space.

In terms of the 16 components of the two twistors Z_1, Z_2, and their complex conjugates, the Liouville one-form Θ in Eq. (1.7) reads

$$\Theta = \frac{i}{2} \left(\omega^\alpha d\pi_\alpha + \overline{\pi}_{\dot\alpha} d\overline{\omega}^{\dot\alpha} - c.c. \right) + \frac{i}{2} \left(\lambda^\alpha d\eta_\alpha + \overline{\eta}_{\dot\alpha} d\overline{\lambda}^{\dot\alpha} - c.c. \right) . \qquad (2.1)$$

[1]We identify TPB and Dirac brackets for particular parametrization of M_8, given by real projective spinors $\widetilde{\eta}_\alpha = \frac{\eta_\alpha}{\overline{\xi}_{\dot\beta}\overline{\eta}^{\dot\beta}}, \widetilde{\sigma}_\alpha = \frac{\sigma_\alpha}{\overline{\xi}_{\dot\beta}\overline{\sigma}^{\dot\beta}}$ (see [6, 7]).

Using Eq. (1.3) and writing

$$z^{\alpha\dot\beta} = x^{\alpha\dot\beta} + iy^{\alpha\dot\beta} \qquad (z^\mu = x^\mu + iy^\mu) , \tag{2.2}$$

one gets:

$$\begin{aligned}
\Theta &= \pi_\alpha \overline{\pi}_{\dot\beta} dx^{\alpha\dot\beta} + iy^{\alpha\dot\beta} \left(\pi_\alpha d\overline{\pi}_{\dot\beta} - \overline{\pi}_{\dot\beta} d\pi_\alpha \right) \\
&\quad + \eta_\alpha \overline{\eta}_{\dot\beta} dx^{\alpha\dot\beta} + iy^{\alpha\dot\beta} \left(\eta_\alpha d\overline{\eta}_{\dot\beta} - \overline{\eta}_{\dot\beta} d\eta_\alpha \right) .
\end{aligned} \tag{2.3}$$

Using the definition (1.9a) of P_μ we obtain

$$\Theta = P_\mu dx^\mu + iy^{\alpha\dot\beta} \left(\pi_\alpha d\overline{\pi}_{\dot\beta} - \overline{\pi}_{\dot\beta} d\pi_\alpha + \eta_\alpha d\overline{\eta}_{\dot\beta} - \overline{\eta}_{\dot\beta} d\eta_\alpha \right) . \tag{2.4}$$

Further, it can be shown that (see (1.11a)–(1.11b))

$$t_{11} = -2y^{\alpha\dot\beta} \pi_\alpha \overline{\pi}_{\dot\beta} , \quad t_{22} = -2y^{\alpha\dot\beta} \eta_\alpha \overline{\eta}_{\dot\beta}, \quad \rho = t_{12} = \overline{t}_{21} = -2y^{\alpha\dot\beta} \eta_\alpha \overline{\pi}_{\dot\beta} , \tag{2.5}$$

where

$$y^{\alpha\dot\beta} = \frac{1}{2f\overline{f}} \left(\rho \pi^\alpha \overline{\eta}^{\dot\beta} + \overline{\rho} \eta^\alpha \overline{\pi}^{\dot\beta} - t_{11} \eta^\alpha \overline{\eta}^{\dot\beta} - t_{22} \pi^\alpha \overline{\pi}^{\dot\beta} \right) , \tag{2.6}$$

and f is given by (1.2). Subsequently,

$$y^{\alpha\dot\beta} \pi_\alpha = \frac{1}{2f} \left(t_{11} \overline{\eta}^{\dot\beta} - \overline{\rho}\, \overline{\pi}^{\dot\beta} \right) , \qquad y^{\alpha\dot\beta} \eta_\alpha = \frac{1}{2f} \left(\rho \overline{\eta}^{\dot\beta} - t_{22} \overline{\pi}^{\dot\beta} \right) , \tag{2.7}$$

and one gets

$$\Theta = P_\mu dx^\mu + \left\{ \frac{i}{2f} \left(t_{11} \overline{\eta}^{\dot\alpha} - \overline{\rho}\, \overline{\pi}^{\dot\alpha} \right) d\overline{\pi}_{\dot\alpha} + \frac{i}{2f} \left(\rho \overline{\eta}^{\dot\alpha} - t_{22} \overline{\pi}^{\dot\alpha} \right) d\overline{\eta}_{\dot\alpha} + c.c. \right\} . \tag{2.8}$$

We see that the composite space-time coordinates x_μ, which have nonvanishing TPB among themselves given by (1.8), enter in the one-form (2.8). Thus, we have arrived at a symplectic formalism with non-commutative space-time coordinates. In principle, one could pursuit the construction of

dynamical models of non-commutative classical mechanics based on the Liouville one-form (2.8). However, one can do better: following recent results [6]–[8] one can define the commutative composite space-time coordinates X_μ by making a shift, $x_\mu \to X_\mu = x_\mu + \Delta x_\mu$, in such a way that the composite relativistic phase space variables (X_μ, P_μ) satisfy the standard relations (1.13). For that purpose we redefine (*cf.* Eq.(1.1))

$$z^{\alpha\dot\beta} \to Z^{\alpha\dot\beta} \;\;=\;\; z^{\alpha\dot\beta} + \Delta\, z^{\alpha\dot\beta} = z^{\alpha\dot\beta} + i\rho\frac{\pi^\alpha \overline\eta^{\dot\beta}}{f\overline f} = X^{\alpha\dot\beta} + iY^{\alpha\dot\beta}\,, \quad (2.9a)$$

where $X^{\alpha\dot\beta} = \frac{1}{2}\sigma_\mu^{\alpha\dot\beta}X^\mu$ and $X^\mu = \mathrm{Re}Z^\mu = \mathrm{Re}(z^\mu + \Delta z^\mu)$ or, explicitly,

$$X^\mu \;\;=\;\; x_\mu + \frac{i}{2f\overline f}(\sigma^\mu)_{\alpha\dot\beta}\left(\rho\,\pi^\alpha\,\overline\eta^{\dot\beta} - \overline\rho\,\eta^\alpha\,\overline\pi^{\dot\beta}\right)\,. \qquad (2.9b)$$

Using the TPB (1.5a) and (1.5b) one can show that the real coordinates (2.9b) commute. Further we get

$$P_\mu d(\Delta x^\mu) \;\;=\;\; P_\mu\left(dX^\mu - dx^\mu\right) =$$

$$=\;\; \frac{1}{2}\left(\pi_\alpha\overline\pi_{\dot\beta} + \eta_\alpha\overline\eta_{\dot\beta}\right)d\left(\frac{i\rho\pi^\alpha\overline\eta^{\dot\beta}}{f\overline f} + c.c.\right)\,, \qquad (2.10)$$

and we arrive at the formula

$$\Theta = P_\mu dX^\mu + \left[\frac{i}{f}\left(\rho\overline\eta^{\dot\alpha} + k\overline\pi^{\dot\alpha}\right)d\overline\eta_{\dot\alpha} + c.c.\right] + \frac{i}{2}t_{11}\left(\frac{d\overline f}{\overline f} - \frac{df}{f}\right)\,. \qquad (2.11)$$

Introducing

$$\sigma^\alpha \;\;=\;\; -\frac{1}{f}\left(\overline\rho\eta^\alpha + k\pi^\alpha\right)\,, \qquad \overline\sigma^{\dot\alpha} = -\frac{1}{\overline f}\left(\rho\overline\eta^{\dot\alpha} + k\overline\pi^{\dot\alpha}\right)\,, \qquad (2.12a)$$

$$e \;\;=\;\; t_{11}\,, \qquad\qquad\qquad \phi = \frac{i}{2}\ln\frac{\overline f}{f}\,, \qquad\qquad (2.12b)$$

one obtains finally

$$\Theta = P_\mu dX^\mu - i\left(\overline{\sigma}^{\dot\alpha} d\overline{\eta}_{\dot\alpha} - c.c.\right) + e d\phi \,. \tag{2.13}$$

The 18 variables $(X_\mu,\ P_\mu,\ \eta_\alpha,\ \overline{\eta}_{\dot\alpha},\ \sigma_\alpha,\ \overline{\sigma}_{\dot\alpha},\ e,\ \phi)$ are not all independent. Comparing the number of degrees of freedom (16 in Eq. (2.1) versus 18 in Eq. (2.13)) one can deduce that the variables occurring in Eq. (2.13) satisfy two constraints. The first one takes the form $(\mathbf{t}^2 = t_1^2 + t_2^2 + t_3^2)$:

$$R_1 \ = \ \sigma_\alpha P^{\alpha\dot\beta}\overline{\sigma}_{\dot\beta} - \mathbf{t}^2 = 0, \quad \mathbf{t}^2 = |\rho|^2 + k^2 \,. \tag{2.14}$$

In order to show explicitly how Eq. (2.14) restricts the eighteen variables of our generalized phase space $(X_\mu, P_\mu, \eta_\alpha, \overline{\eta}_{\dot\alpha}, \sigma_\alpha, \overline{\sigma}_{\dot\alpha}, e, \phi)$ we observe that

$$k = \overline{\eta}^{\dot\alpha}\,\overline{\sigma}_{\dot\alpha} \ = \ \eta^\alpha\,\sigma_\alpha = \overline{k}\,, \tag{2.15}$$

$$\rho = \overline{\pi}_{\dot\alpha}\overline{\sigma}^{\dot\alpha} = -\frac{1}{\overline{f}}\eta_\alpha P^{\alpha\dot\beta}\overline{\sigma}_{\dot\beta}\,, \qquad \overline{\rho} = \pi_\alpha \sigma^\alpha = -\frac{1}{f}\sigma_\alpha\,P^{\alpha\dot\beta}\overline{\eta}_{\dot\beta}\,, \tag{2.16a}$$

$$|\rho|^2 \ = \ \frac{2}{P^2} P^{\alpha\dot\beta} P^{\gamma\dot\delta}\,\sigma_\alpha\,\overline{\eta}_{\dot\beta}\,\overline{\sigma}_{\dot\delta}\,\eta_\gamma\,, \tag{2.16b}$$

where

$$\pi_\alpha = -\frac{1}{f}\,P_{\alpha\dot\beta}\,\overline{\eta}^{\dot\beta}\,, \qquad f = \frac{1}{\sqrt{2}}\,P\,e^{i\phi}\,, \qquad P \equiv (P^\mu P_\mu)^{\frac{1}{2}}\,. \tag{2.17}$$

Taking into account Eq. (2.15), the reality of the variable k produces the second constraint equation

$$R_2 \ = \ \eta_\alpha\,\sigma^\alpha - \overline{\eta}_{\dot\alpha}\,\overline{\sigma}^{\dot\alpha} = 0\,. \tag{2.18}$$

We point out that if we employ the composite formulae (1.9a) and (2.12a) both R_1 and R_2 are identically zero in terms of the two-twistor variables (1.4).

If we select as generalized momenta the set of the nine commuting variables $(P_\mu, \eta_\alpha, \overline{\eta}_{\dot\alpha}, e)$ they satisfy also the following constraint $(P^2 \equiv P_{\alpha\dot\beta}P^{\alpha\dot\beta} = 2f\overline{f})$

$$\eta_\alpha P^{\alpha\dot\beta}\overline\eta_{\dot\beta} \;=\; \frac{1}{2}P^2\,. \tag{2.19}$$

It can be shown, however, that the constraint (2.19) follows from the constraints (2.14) and (2.18). Therefore, using (2.19) the mass m can be defined by the following subsidiary condition:

$$R_3 \;=\; \eta_\alpha P^{\alpha\dot\beta}\overline\eta_{\dot\beta} - \frac{1}{2}m^2 = 0\,. \tag{2.20}$$

Much in the same way as the constraint (2.19) provides the mass Casimir P^2, the constraint (2.14) defines the square of the relativistic spin operator. Its numerical value provides the fourth constraint:

$$R_4 \;=\; \sigma_\alpha\, P^{\alpha\dot\beta}\overline\sigma_{\dot\beta} - s(s+1) = 0\,. \tag{2.21}$$

Indeed, from the formula (1.9b) defining the Pauli-Lubański relativistic spin fourvector W^μ it follows that

$$k = \frac{1}{2|f|^2}\left(\eta_\alpha\overline\eta_{\dot\beta} - \pi_\alpha\,\overline\pi_{\dot\beta}\right)W^{\alpha\dot\beta}\,,$$

$$\rho = -\frac{1}{|f|^2}\,\pi_\alpha\,\overline\eta_{\dot\beta}W^{\alpha\dot\beta}\,, \qquad \overline\rho = -\frac{1}{|f|^2}\,\eta_\alpha\,\overline\pi_{\dot\beta}W^{\alpha\dot\beta}\,. \tag{2.22}$$

Using Eq. (1.9b), one obtains $(W^2 \equiv W_{\alpha\dot\beta}W^{\alpha\dot\beta} = W_\mu W^\mu)$:

$$\mathbf{t}^2 \;=\; -\frac{1}{2f\overline f}\,W_{\alpha\dot\beta}W^{\alpha\dot\beta} = -\frac{1}{P^2}\,W^2\,. \tag{2.23}$$

Because $\{P_\mu,\rho\} = \{P_\mu,k\} = 0$ one gets $\{P_\mu,\sigma_\alpha\} = 0$ and (Eqs. (2.20), (2.21))

$$\left\{\mathbf{t}^2, P^2\right\} \;=\; 0\,. \tag{2.24}$$

We see therefore that the constraints $R_3 = R_4 = 0$ on the constrained 16-dimensional generalized phase space take the form $(m \geq 0; s = 0, \frac{1}{2}, 1, \ldots)$, on account of Eqs. (2.19) and (2.14),

$$P^2 = m^2 \,, \qquad \mathbf{t}^2 = s(s+1) \,, \tag{2.25}$$

and provide the mass and spin values characterizing a relativistic particle. The electric charge e_0 is defined by the fifth subsidiary condition

$$R_5 = e - e_0 = 0 \,, \tag{2.26}$$

where e is given by (2.12b).

3 Relativistic particle model from the two-twistor framework

Looking at the Liouville one-form (2.13), with all 18 coordinates X_μ, P_μ, Y_k now treated as primary, we propose the following action for a charged, massive relativistic particle with spin

$$
\begin{aligned}
S \;=\; & \int d\tau \mathcal{L} = \int d\tau \left[P_\mu \dot{X}^\mu + i \left(\sigma^\alpha \dot{\eta}_\alpha - \overline{\sigma}^{\dot\alpha} \dot{\overline{\eta}}_{\dot\alpha} \right) + e\dot{\phi} \right. \\
& \left. + \lambda_1 R_1 + \lambda_2 R_2 + \xi_1 (P^2 - m^2) + \xi_2 (\mathbf{t}^2 - s(s+1)) + \xi_3 (e - e_0) \right] \,,
\end{aligned}
\tag{3.1}
$$

where $e.g.$ $\dot{\overline{\eta}}_{\dot\alpha} = d\overline{\eta}_{\dot\alpha}/d\tau$ and the λ's and ξ's are Lagrange multipliers.

The canonical Poisson brackets determined by the action (3.1) will be denoted by $\{\,\cdot\,,\,\cdot\,\}_C$. They are

$$\{X_\mu, X_\nu\}_C = 0 \,, \quad \{P_\mu, P_\nu\}_C = 0 \,, \quad \{P_\mu, X_\nu\}_C = \eta_{\mu\nu} \,, \tag{3.2a}$$

$$\left\{\eta_\alpha, \sigma^\beta\right\}_C = i\delta_\alpha{}^\beta \,, \qquad \left\{\overline{\eta}_{\dot\alpha}, \overline{\sigma}^{\dot\beta}\right\}_C = -i\delta_{\dot\alpha}{}^{\dot\beta} \,, \tag{3.2b}$$

$$\{e, \phi\}_C = 1 \,, \tag{3.2c}$$

all others being zero. The five constraints described by the action (3.1) via the Lagrange multipliers split into a pair, (2.14) and (2.18), of second class constraints, reducing the number of degrees of freedom from 18 to 16,

and three first class constraints, Eqs. (2.20)–(2.21) (or equivalently (2.25)) plus Eq. (2.26). These three first class constraints define the mass, spin and electric charge values, which further reduce the number of degrees of freedom from 16 to 10.

Let us observe that in order to obtain the consistency of the Poisson structure with the second class constraints $R_1 = R_2 = 0$, the canonical PB (3.2a)–(3.2c) have to be replaced by Dirac brackets given by

$$
\begin{aligned}
\{Y_k, Y_l\}_D &= \{Y_k, Y_l\}_C + \{Y_k, R_1\}_C \frac{1}{\{R_1, R_2\}_C} \{R_2, Y_l\}_C \\
&\quad - \{Y_k, R_2\}_C \frac{1}{\{R_1, R_2\}_C} \{R_1, Y_l\}_C \,,
\end{aligned}
\tag{3.3}
$$

where

$$
\{R_1, R_2\}_C = -2i\sigma_\alpha\, P^{\alpha\dot\beta}\overline\sigma_{\dot\beta} \,.
\tag{3.4}
$$

For the spin sector variables $(\eta_\alpha, \overline\eta_{\dot\alpha}, \sigma_\alpha, \overline\sigma_{\dot\alpha})$ one gets

$$
\{\eta_\alpha, \eta_\beta\}_D = \left\{\overline\eta_{\dot\alpha}, \overline\eta_{\dot\beta}\right\}_D = \left\{\eta_\alpha, \overline\eta_{\dot\beta}\right\}_D = 0 \,,
\tag{3.5a}
$$

$$
\left\{\sigma^\alpha, \sigma^\beta\right\}_D = \frac{-i}{P^2}(P^{\alpha\dot\gamma}\overline\eta_{\dot\gamma}\sigma^\beta - P^{\beta\dot\gamma}\overline\eta_{\dot\gamma}\sigma^\alpha) \,,
\tag{3.5b}
$$

$$
\left\{\sigma^\alpha, \overline\sigma^{\dot\beta}\right\}_D = \frac{-i}{P^2}(P^{\alpha\dot\gamma}\overline\eta_{\dot\gamma}\overline\sigma^{\dot\beta} + \eta_\gamma P^{\gamma\dot\beta}\sigma^\alpha) \,,
\tag{3.5c}
$$

$$
\left\{\overline\sigma^{\dot\alpha}, \overline\sigma^{\dot\beta}\right\}_D = \frac{i}{P^2}(\eta_\gamma P^{\gamma\dot\alpha}\overline\sigma^{\dot\beta} - \eta_\gamma P^{\gamma\dot\beta}\overline\sigma^{\dot\alpha}) \,,
\tag{3.5d}
$$

$$
\{\eta_\alpha, \sigma^\beta\}_D = i\delta_\alpha{}^\beta - \frac{i}{P^2}\eta_\alpha P^{\beta\dot\gamma}\overline\eta_{\dot\gamma} \,,
\tag{3.5e}
$$

$$
\left\{\overline\eta_{\dot\alpha}, \overline\sigma^{\dot\beta}\right\}_D = -i\,\delta_{\dot\alpha}^{\dot\beta} + \frac{i}{P^2}\eta_\gamma P^{\gamma\dot\beta}\overline\eta_{\dot\alpha} \,,
\tag{3.5f}
$$

$$
\left\{\eta_\alpha, \overline\sigma^{\dot\beta}\right\}_D = \frac{i}{P^2}\eta_\alpha\eta_\gamma P^{\gamma\dot\beta} \,.
\tag{3.5g}
$$

The relations (3.5a)–(3.5g) provide the Dirac bracket structure for the six degrees of freedom of the spin phase space $(\eta_\alpha, \overline\eta_{\dot\alpha}, \sigma_\alpha, \overline\sigma_{\dot\alpha})$, consistent with the constraints $R_1 = 0 = R_2$. Further, we have

$$
\left\{\eta_\alpha, P^{\beta\dot\gamma}\right\}_D = \left\{\overline\eta_{\dot\alpha}, P^{\beta\dot\gamma}\right\}_D = \left\{\sigma_\alpha, P^{\beta\dot\gamma}\right\}_D = \left\{\overline\sigma_{\dot\alpha}, P^{\beta\dot\gamma}\right\}_D = 0 \,,
\tag{3.6}
$$

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but

$$\{\eta_\alpha, X^{\beta\dot\gamma}\}_D = \eta_\alpha \frac{\widetilde{P}^{\beta\dot\gamma}}{P^2}, \qquad \{\sigma_\alpha, X^{\beta\dot\gamma}\}_D = -\sigma_\alpha \frac{\widetilde{P}^{\beta\dot\gamma}}{P^2}, \qquad (3.7)$$

where

$$\widetilde{P}^{\beta\dot\gamma} = P^{\beta\dot\gamma} \frac{k^2}{\rho\overline{\rho} + k^2} - \eta^\beta \overline{\eta}^{\dot\gamma}. \qquad (3.8)$$

Because ϕ and e have vanishing Poisson brackets with R_1 and R_2 one obtains

$$\{e, \phi\}_D = 1. \qquad (3.9)$$

Finally, since $\{R_{1,2}, P\}_C = 0 = \{R_2, X^{\alpha\dot\beta}\}_C$, the Dirac brackets in the relativistic phase sector (X_μ, P_ν) coincide with the canonical ones given by Eq. (3.2a).

The canonical Poisson brackets (3.2a) are useful if we calculate from the action (3.1) the Noether charge $\Sigma_{\mu\nu}$, describing the generators of the Lorentz transformations in the spin sector $i.e.$, the spin part of the relativistic angular momentum. One obtains, using spinorial notation (see also [8])

$$\Sigma_{\mu\nu} = \frac{1}{2i}\sigma_\mu^{\alpha\dot\alpha}\sigma_\nu^{\beta\dot\beta}(\sigma_{(\alpha}\,\eta_{\beta)}\,\epsilon_{\dot\alpha\dot\beta} - \overline{\sigma}_{(\dot\alpha}\,\overline{\eta}_{\dot\beta)}\epsilon_{\alpha\beta}) = -\Sigma_{\nu\mu}, \qquad (3.10)$$

where the symmetrization is with unit weight. From (3.2b) follows that[2]

$$\{\Sigma_{\mu\nu}, \Sigma_{\rho\tau}\}_C = \eta_{\mu\tau}\Sigma_{\rho\nu} - \eta_{\mu\rho}\Sigma_{\nu\tau} + \eta_{\rho\nu}\Sigma_{\mu\tau} - \eta_{\nu\tau}\Sigma_{\mu\rho}. \qquad (3.11)$$

It is easy to show that the Lorentz spin generators (3.10) imply the following values of the two Lorentz Casimirs:

$$C_1 = \frac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu} = \frac{1}{2}\left[(\eta_\alpha\,\sigma^\alpha)^2 + (\overline{\eta}_{\dot\alpha}\,\overline{\sigma}^{\dot\alpha})^2\right] = k^2, \qquad (3.12a)$$

$$C_2 = \frac{1}{8}\Sigma_{\mu\nu}\epsilon^{\mu\nu\rho\tau}\Sigma_{\rho\tau} = 0, \qquad (3.12b)$$

where Eq. (2.15) has been used. We notice that $C_2 \neq 0$ requires having n-twistor coordinates with $n > 2$.

_____

[2]Using that the total generators of the Lorentz transformations $M^{\mu\nu} = 2P^{[\mu}X^{\nu]} + \Sigma^{\mu\nu}$ fulfil the Lorentz algebra and that they have vanishing PB with the two second class constraints R_1, R_2, one can also show that $\{\Sigma_{\mu\nu}, \Sigma_{\rho\tau}\}_D = \{\Sigma_{\mu\nu}, \Sigma_{\rho\tau}\}_C$.

In order to quantize the model described by the action (3.1), the quantum counterparts of the first class constraints $R_3 = R_4 = R_5 = 0$ are needed. One can proceed in two different ways.

i) The covariant formulation consists in treating the three first class constraints as conditions on quantum states which lead to wave equations. For this aim one considers the differential realization of the Dirac brackets (3.5a)–(3.7) on the generalized momentum space $(P_\mu, \eta_\alpha, \overline{\eta}_{\dot\alpha}, \phi)$ that provides the Schrödinger representation for the first-quantized theory.

ii) Another way is to look at the first class constraints $R_3 = R_4 = R_5 = 0$ as generators of three local symmetries, and to fix the gauge of the corresponding local degrees of freedom. In such a way the three gauge-fixing conditions plus the constraints $R_3 = R_4 = R_5 = 0$ provide six additional second class constraints. These reduce the 16 degrees of freedom of two-twistor space to the "physical" ten-dimensional generalized phase space. In this formulation the gauge-fixing conditions break necessarily the Lorentz invariance and lead to a noncovariant formulation of the first-quantized theory in the Heisenberg picture.

Both methods of quantization are under consideration by the present authors.

4   Conclusions

The main aim of this paper was to show how to move from the two-twistor geometry to a generalized space-time description of relativistic particle mechanics. In order to introduce our particle model action, Eq. (3.1), we used the symplectic potential (Liouville one-form) (2.13) obtained from the composite nature of the variables $(X_\mu, P_\mu, \sigma_\alpha, \overline{\sigma}_{\dot\alpha}, e, \phi)$; only $\eta_\alpha, \overline{\eta}_{\dot\alpha}$ remain as primary twistor coordinates. We stress again that in such a framework both the fourmomentum P_μ (Eq. (1.9a)) as well as the real Minkowski space-time coordinates X_μ (Eq. (2.9b)) are composite. Nevertheless, in the action (3.1) all the eighteen variables X_μ, P_μ, Y_k ($k = 1, \ldots, 10$) are taken as primary ones and only the presence of the constraints $R_1 = R_2 = 0$ exhibits their twistorial origin. On our sixteen-dimensional generalized phase space $\{X_\mu, P_\mu, M_8\}$ one can introduce two Poisson structures consistent with the pair of second class constraints: one $\{\,\cdot\,,\,\cdot\,\}$, induced by the fundamental TPB (1.5a) and (1.5b) and applied to the composite twistor formulae, and a second one $\{\,\cdot\,,\,\cdot\,\}_D$, given by the Dirac brackets (3.3) and further explicitly calculated in (3.5a)–

(3.9).

We would like to report here an important calculational result: for the 8-dimensional parametrization of M_8, given by the projective real spinors $\widetilde{\eta}_{\dot\alpha} = \frac{\eta_\alpha}{\xi_{\dot\alpha}\overline{\eta}^{\dot\alpha}}$, $\widetilde{\sigma}_\alpha = \frac{\sigma_\alpha}{\xi_{\dot\alpha}\overline{\sigma}^{\dot\alpha}}$ (ξ_α is a constant spinor; see [6, 7]) supplemented by the pair of variables (e, φ), these two Poisson structures are the same.

In this paper we have limited ourselves to the classical theory and provided an outline on how to construct the first-quantized theory. Clearly, the first-quantized theory is important because it provides the description of a class of relativistic free fields, those with spin generators restricted by the constraint (3.12b). Our final aim is to obtain the covariant description of all massive Wigner representations of the Poincaré group as solutions of the first-quantized free particle model (3.1), or of its generalization to a three-twistor space. Finally, we recall that our two-twistor framework also provides, besides the spin description, the canonical pair of variables (see (3.9)) describing a $U(1)$ internal gauge degree of freedom as well as the electric charge.

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